DUALITY IN THE FORMULAS OF SPHERICAL TRIGONOMETRY.

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It was Moebius* who first called attention to the fact that if we use the supplements of the angles of a spherical triangle instead of the angles themselves, the formulas of Spherical Trigonometry arrange themselves in pairs, either one of a pair being the dual of the other. While this very important notion is now generally employed in advanced treatises dealing with Geometry on the Sphere, I have been unable to discover any writer who has carried out this fruitful idea of duality to its logical conclusion, and that is to apply it to the solution of spherical triangles in a first course in Spherical Trigonometry.

The purpose of this paper is to call the attention of teachers to the many practical advantages attending the use of this property of duality in the formulas of Spherical Trigonometry in teaching the elements of Spherical Trigonometry.

Moebius proved his results for any spherical triangle. For the sake of brevity, however, I shall follow the ordinary practice in a first course and consider such spherical triangles only whose parts are less than 180°.

Given any relation involving one or more of the sides a, b, c, and the angles A, B, C, of any spherical triangle. Now the polar triangle (whose sides are denoted by a', b', c', and angles by A', B', C') is also a spherical triangle and the given relation must hold true for it also; that is, the given relation applies to the polar triangle if accents are placed on the letters representing the sides and angles. Thus, the First Law of Cosines, as usually given, is

$$\cos a = \cos b \cos c + \sin b \sin c \cos A, \tag{1}$$

or, for the polar triangle,

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \sin A'. \tag{2}$$

But $a' = 180° - A$, $b' = 180° - B$, $c' = 180° - C$, $A' = 180° - a$.

Substituting these in (2), we get

$$\begin{aligned}\cos(180° - A) = {} & \cos(180° - B) \cos(180° - C) \\ & + \sin(180° - B) \sin(180° - C) \cos(180° - a),\end{aligned}$$

or (the Second Law of Cosines),

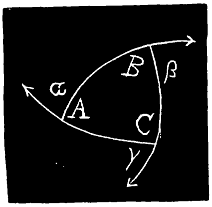
*Moebius: *Ueber eine neue Behandlungsweise der analytischen Sphaerik*, 1846. *Entwicklung der Grundformeln der Trigonometrie in grosstmoeglicher Algemeinheit*, 1860. See *Gesammelte Werke II.*

(3) $$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

which involves the sides and angles of the original triangle. Hence,

THEOREM. *In any relation between the parts of a spherical triangle each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.*

Let the supplements of the angles of the triangle be denoted by α, β, γ; *i, e.* α, β, γ are the exterior angles of the triangle. Then



$$\alpha = 180^\circ - A, \ \beta = 180^\circ - B, \ \gamma = 180^\circ - C,$$
$$\text{or, } A = 180^\circ - \alpha, \ B = 180^\circ - \beta, \ C = 180^\circ - \gamma.$$

When we apply the above theorem to a relation involving the sides and the supplements of the angles of a triangle, we, in fact,

replace a by α $(=180^\circ - A)$,
replace b by β $(=180^\circ - B)$,
replace c by γ $(=180^\circ - C)$,
replace α $(=180^\circ - A)$ by $180^\circ - (180^\circ - a) = a$,
replace β $(=180^\circ - B)$ by $180^\circ - (180^\circ - b) = b$,
replace γ $(=180^\circ - C)$ by $180^\circ - (180^\circ - c) = c$,

or, what amounts to the same thing, *we interchange the Greek and Roman letters.* For instance, substitute

$$A = 180^\circ - \alpha, \ B = 180^\circ - \beta, \ C = 180^\circ - \gamma \text{ in (1).}$$

This gives us the First Law of Cosines in the new form

(4) $$\cos a = \cos b \cos c - \sin b \sin c \cos \alpha.$$

Similarly, we may get

(5) $$\cos b = \cos c \cos a - \sin c \sin a \cos \beta,$$
(6) $$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma.$$

If we now make the substitutions indicated above in (4), (5), (6); that is, interchange the Greek and Roman letters, we get the following new form of the Second Law of Cosines:

(7) $$\cos \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a,$$
(8) $$\cos \beta = \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b,$$
(9) $$\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c;$$

that is, we have derived three new relations between the sides and the sup-

plements of the angles of the triangle. Summing up the results of our discussion, we may then state the following

PRINCIPLE OF DUALITY ON THE SPHERE. *If the sides of a spherical triangle be denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters α, β, γ, then from any given formula involving any of these six parts, we may write down a dual formula by simply interchanging the corresponding Greek and Roman letters.*

For the sake of comparison the formulas that are commonly used in the solution of oblique spherical triangles are given in the first column below, while the corresponding new forms are written in the second column.

First Law of Cosines.

$\cos a=\cos b\cos c+\sin b\sin c\cos A$	$\cos a=\cos b\cos c-\sin b\sin c\cos\alpha$
$\cos b=\cos c\cos a+\sin c\sin a\cos B$	$\cos b=\cos c\cos a-\sin c\sin a\cos\beta$
$\cos c=\cos a\cos b+\sin a\sin b\cos C$	$\cos c=\cos a\cos b-\sin a\sin b\cos\gamma$

Second Law of Cosines.

$\cos A=-\cos B\cos C+\sin B\sin C\cos a$	$\cos\alpha=\cos\beta\cos\gamma-\sin\beta\sin\gamma\cos a$
$\cos B=-\cos C\cos A+\sin C\sin A\cos b$	$\cos\beta=\cos\gamma\cos\alpha-\sin\gamma\sin\alpha\cos b$
$\cos C=-\cos A\cos B+\sin A\sin B\cos c$	$\cos\gamma=\cos\alpha\cos\beta-\sin\alpha\sin\beta\cos c$

The formulas for the functions of half the supplements of the angles of a spherical triangle in terms of its sides may be derived in the same manner as the formulas for the functions of half the angles of a triangle in terms of the sides are usually derived. This gives

$\sin\frac{1}{2}A=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$	$\sin\frac{1}{2}\alpha=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}}$
$\cos\frac{1}{2}A=\sqrt{\dfrac{\sin s\sin(s-a)}{\sin b\sin c}}$	$\cos\frac{1}{2}\alpha=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}$
$\tan\frac{1}{2}A=\sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}$	$\tan\frac{1}{2}\alpha=\sqrt{\dfrac{\sin s\sin(s-c)}{\sin(s-b)\sin(s-c)}}$

By permuting the letters in cyclical order, we get the remaining two sets of formulas of six each.

The usual method is now to derive the formulas for the functions of the half sides in terms of the angles (shown below in the first column) in like manner. By the use of the Principle of Duality, however, we get the corresponding formulas at once (as shown in the second column) by simply interchanging the Greek and Roman letters in the second column above.

$$\sin\tfrac{1}{2}a=\sqrt{\frac{-\cos A\cos(S-A)}{\cos B\cos C}}\qquad \sin\tfrac{1}{2}\alpha=\sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin\beta\sin\gamma}}$$

$$\cos\tfrac{1}{2}a=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}}\qquad \cos\tfrac{1}{2}\alpha=\sqrt{\frac{\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\beta\sin\gamma}}$$

$$\tan\tfrac{1}{2}a=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}\qquad \tan\tfrac{1}{2}\alpha=\sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin(\sigma-\beta)\sin(\sigma-\gamma)}}$$

where $s=\frac{1}{2}(a+b+c)$, $S=\frac{1}{2}(A+B+C)$, $\sigma=\frac{1}{2}(\alpha+\beta+\gamma)$.

Instead of the above two sets of formulas it is sometimes more convenient to use the following two sets when solving triangles.

$$\tan r=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}\qquad \tan\tfrac{1}{2}d=\sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$$

$$\tan\tfrac{1}{2}A=\frac{\tan r}{\sin(s-a)}\qquad \tan\tfrac{1}{2}\alpha=\frac{\sin(s-a)}{\tan\frac{1}{2}d}$$

etc., where d is the diameter of the inscribed circle, and

$$\tan R=\sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}}$$

$$\tan\tfrac{1}{2}a=\tan R\cos(S-A)$$

$$\tan\tfrac{1}{2}\delta=\sqrt{\frac{\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\sigma}}$$

$$\tan\tfrac{1}{2}\alpha=\frac{\sin(\sigma-\alpha)}{\tan\frac{1}{2}\delta}$$

etc., where δ is the supplement of the diameter of the circumscribed circle.

Napier's Analogies in the old and new forms are shown below.

$$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\qquad \tan\tfrac{1}{2}(\alpha+\beta)=-\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma$$

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\qquad \tan\tfrac{1}{2}(\alpha-\beta)=-\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma$$

$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c\qquad \tan\tfrac{1}{2}(a+b)=-\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c$$

$$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c\qquad \tan\tfrac{1}{2}(a-b)=-\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c$$

In the case of the Law of Sines the usual form is,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

If we use the supplements of the angles this becomes

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}.$$

If we apply the Principle of Duality to the last form there results

$$\frac{\sin \alpha}{\sin a}=\frac{\sin \beta}{\sin b}=\frac{\sin \gamma}{\sin c};$$

hence, the *Law of Sines goes over into itself.*

In what follows we have in the first column the usual ten formulas used in the solution of right triangles, in terms of α, β, γ. When we apply the Principle of Duality to these we get the formulas in the second column, and these turn out to be formulas for the solution of quadrantal triangles.

	Right Triangle.	*Quadrantal Triangle*
(1)	$\cos c=\cos a \cos b$	$\cos\gamma=\cos\alpha\cos\beta$
(2)	$\sin a=\sin c \sin\alpha$	$\sin\alpha=\sin\gamma\sin a$
(3)	$\sin b=\sin c\sin\beta$	$\sin\beta=\sin\gamma\sin b$
(4)	$\cos\alpha=-\cos a\sin\beta$	$\cos a=-\cos\alpha\sin b$
(5)	$\cos\beta=-\cos b\sin\alpha$	$\cos b=-\cos\beta\sin a$
(6)	$\cos\alpha=-\tan b\cot c$	$\cos a=-\tan\beta\cot\gamma$
(7)	$\cos\beta=-\tan a\cot c$	$\cos b=-\tan\alpha\cot\gamma$
(8)	$\sin b=-\tan a\cot\alpha$	$\sin\beta=-\tan\alpha\cot a$
(9)	$\sin a=-\tan b\cot\beta$	$\sin\alpha=-\tan\beta\cot b$
(10)	$\cos c=\cot\alpha\cot\beta$	$\cos\gamma=\cot a\cot b$

Von Staudt called the expression

$$D=\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c-2\cos a\cos b\cos c}$$

the *sine of the spherical triangle,* and

$$\triangle=\sqrt{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma}$$

the *sine of the polar triangle,* because of the relation

$$\frac{\sin a}{\sin \alpha}=\frac{\sin b}{\sin \beta}=\frac{\sin c}{\sin \gamma}=\frac{D}{\triangle}.$$

Other authors have called D the *first staudtian* and $\triangle$ the *second staudtian.* We see that either one goes over into the other by the application of the Principle of Duality. This is also shown by the relations

$$D=\sin b \sin c \sin \alpha=\sin c \sin a \sin \beta=\sin a \sin b \sin \gamma,$$
$$\triangle=\sin \alpha \sin \beta \sin a=\sin \beta \sin \gamma \sin b=\sin \gamma \sin \alpha \sin c.$$

Let V denote the volume of the tetrahedron whose vertices are at A, B, C, the vertices of the spherical triangle and O, the center of the sphere. Then it may be shown that

$$6V=r^3D,$$

where r is the radius of the sphere. If we apply the Principle of Duality to this expression we get

$$6V'=r^3\triangle,$$

where V' is the volume of the tetrahedron corresponding to the polar triangle A', B', C'.

From a theoretical standpoint the great advantages derived from using the Principle of Duality instead of the usual methods are so apparent that there is scarcely any room for argument. Nearly one-half of the work usually required in deriving the standard formulas is done away with and the resulting formulas are more easily memorized. But, the question now naturally arises, are these new formulas well adapted to numerical calculations? By actual experience in solving a large number of spherical triangles, I have found that there is little or no difference in the amount of labor involved. In the case of given angles we must of course first find their supplements before applying the new formulas, and in the case of required anggles we must take the supplements of the angles found from the tables. On the other hand I have found that on account of the duality of the new formulas the detail of the work is simplified and there is less liability of making mistakes. Below is found a solution of a spherical triangle, the new formulas being used.

Example. Given $A=70°$, $B=131° 10'$, $C=94° 50'$; find a, b, c.

$\alpha=180°-A=110°$	To find $\log\tan\frac{1}{2}\delta$
$\beta=180°-B=48°\ 50'$	$\log\sin(\sigma-\alpha)=9.3179$
$\gamma=180°-C=85°\ 10'$	$\log\sin(\sigma-\beta)=9.9810$
$2\sigma=244°$	$\log\sin(\sigma-\gamma)=9.7778$
$\sigma=122°$	$\text{colog}\sin\sigma=0.0716$
$\sigma-\alpha=12°$	$2\,\vert\,9.1483$
$\sigma-\beta=73°\ 10'$	$\log\tan\frac{1}{2}\delta=9.5742$
$\sigma-\gamma=36°\ 50'$	

$\log\sin(\sigma-\alpha)=9.3179$	$\log\sin(\sigma-\beta)=9.9810$	$\log\sin(\sigma-\gamma)=9.7778$
$\log\tan\frac{1}{2}\delta=9.5742$	$\log\tan\frac{1}{2}\delta=9.5742$	$\log\tan\frac{1}{2}\delta=9.5742$
$\log\tan\frac{1}{2}a=9.7437$	$\log\tan\frac{1}{2}b=0.4068$	$\log\tan\frac{1}{2}c=0.2036$
$\frac{1}{2}a=39°$	$\frac{1}{2}b=68°\ 36'$	$\frac{1}{2}c=57°\ 58'$
$a=58°$	$b=137°\ 12'$	$c=115°\ 56'$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

308. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Find the conditions that the roots of $x^2+px+q=0$ may not lie between -1 and $+1$.

I. Solution by J. A. CAPARO, University of Notre Dame, Notre Dame, Ind.

In the most general case let one of the roots be $+a$ and the other $-b$, then:

$$(x-a)(x+b)=0 \text{ or } x^2+x(b-a)-ab=0.$$

Comparing with $x^2+px+q=0$, $p=b-a$, $q=-ab$.

The conditions that the roots shall not lie between $+1$ and -1 are:

$$+a>1\ldots(1);\quad +b>1\ldots(2).$$

Multiplying, $ab>1$, but $ab=-q$, therefore $-q>1$ or $q<-1$. Also from (1), (2), $a-1>0$, $b+1>2$.

Multiplying, $ab-b+a-1>0$, or $-ab+(b-a)+1<0$, or $+q+p<-1$. The required conditions then are: $q<-1$ and $q+p<-1$.